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# ON THICKNESS AND DECOMPOSABILITY OF ABELIAN $p$-GROUPS

BY

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ABSTRACT

We construct essentially finitely indecomposable abelian $p$-groups that are not thick, i.e., admit a non-small homomorphism into a $\Sigma$-cyclic $p$-group.

All groups in this paper are separable abelian $p$-groups, $p$ a fixed but arbitrary prime integer, and our notations are standard as in [FuI/II]. First we would like to fix some notation and recall a few definitions. Let $B = \bigoplus_{n<\omega} \mathbb{Z}(p^n)$ be the direct sum of cyclic groups of order $p^n$ and $\bar{B}$ the torsion-completion of $B$. If $G$, $H$ are $p$-groups, let $\mathrm{Hom}_s(G,H)$ be the group of all small homomorphisms $\varphi$ from $G$ into $H$, i.e. for all $n<\omega$ there is $m<\omega$ such that $\varphi((p^mG)[p^n]) = 0$.

A $p$-group $G$ is called **thin** if Hom $(\bar{B}, G)$ =$\mathrm{Hom}_s(\bar{B}, G)$ and $G$ is called **thick** if Hom $(G, B)$ =$\mathrm{Hom}_s(G, B)$. (The original definitions of "thin" and "thick" look a little bit more general but are equivalent to ours.) A group $H$ is **essentially** (finitely) **indecomposable** (ei, efi respectively) if whenever $H = \bigoplus_{i\in I} H_i$, all except one (respectively, all but finitely many) of the $H_i$'s are uniformly bounded by some $p^n$.

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It is easy to see that any thick group is efi. The main purpose of this paper is to construct groups that are efi but not thick. (This answers in the negative a question raised by Irwin some twenty years ago.) We would like to mention that [BW] contains the result that $G$ is thick if and only if: $G$ is efi and $K[p]$ is purifiable for every subgroup $K \subseteq G$ with $G/K$ $\Sigma$-cyclic. Thus our counterexamples will be efi groups that are not pure complete.

Any reader familiar with the results on realizations of rings as endomorphism rings of $p$-groups may expect to find a suitable example in the plethora of groups with prescribed endomorphism rings. Let us consider the $p$-groups $G$ constructed in [CG] for example: Start with a ring $R$ whose additive group $R^+$ is the completion (in the $p$-adic topology) of a free $p$-adic module. Let $\mathcal{N}$ be a system of right ideals of $R$ and equip $R$ with the topology $\tau$ induced by $p^nR+N$, for $n < \omega$ and $N \in \mathcal{N}$. Then there are separable abelian $p$-groups $G$ such that $\operatorname{End} G = R \bigoplus \operatorname{Ines} G$ where the topology $\tau$ on $R$ coincides with the finite topology of $R \subseteq \operatorname{End} G$, cf. [CG]. Here $\operatorname{Ines} G = \operatorname{Hom}_s(G,G)$ if $\mathcal{N} = \{0\}$, and $\operatorname{Ines} G$ properly contains $\operatorname{Hom}_s(G,G)$ if $\mathcal{N} \neq \{0\}$. In the latter case one doesn't know enough to decide if $G$ is thick and/or efi. If $\mathcal{N} = \{0\}$, then $G$ contains a pure $\sum$-cyclic $R$-module of the form

$$S = \bigoplus_{\lambda}(\bigoplus_{n<\omega} R/p^nR),$$

$\lambda$ an uncountable cardinal, and $R \subseteq \operatorname{End} G$ operates as right multiplication on $S$. (Recall that we still refer to the $p$-groups $G$ constructed in [CG].) Suppose that $G$ is not thick. Then there exists a non-small homomorphism $\varphi\colon G \to B$ and we may assume $B \subseteq S \subseteq G$, i.e. $\varphi = r + \sigma \in \operatorname{End} G$, $r \in R$, $\sigma \in \operatorname{Ines} G = \operatorname{Hom}_s(G,G)$. Since $\varphi$ is not small we have $r \neq 0$ and $r \notin p^nR$ for some $n < \omega$. Thus

$$B \supseteq \varphi(G) \supseteq \varphi(p^mG[p^n]) = (p^mG[p^n])r \supseteq (p^mS[p^n])r \supseteq \bigoplus_{\lambda}((p^mR)/(p^{m+n}R))r$$

and $p^mRr \not\subseteq p^{m+n}R$. This is absurd since $B$ is countable and $\lambda$ uncountable. This shows that all $p$-groups constructed in [CG] with $\mathcal{N} = \{0\}$ are thick, and they are also thin by construction, cf. [CG]. Again, if $\mathcal{N} \neq \{0\}$ then $\operatorname{Ines} G \supset \operatorname{Hom}_s(G,G)$ and undesired idempotents may sneak into $\operatorname{End} G$. Thus we will not find a non-thick efi group in [CG]! If $G$ is not thick we may assume that $B \subseteq G$ and there is a non-small $\varphi = r + \sigma \in \operatorname{End} G$ with $\sigma(G) = B$.

Therefore, in order to construct a group $G$ that is efi or ei but not thick, one has to be less ambitious in controlling End$(G)$. We will use the fact that the action of End$(G)$ on the socle $G[p]$ of $G$ can be used to decide the (non)existence of direct summands of $G$. Here a recent paper [DG] points in the right direction. We quote the main result of [DG] as

THEOREM A: *([DG]) Let $R$ be a ring with $1 \in R$, $\kappa$ a cardinal and $\bigoplus_\kappa J_p \subseteq R^+ \subseteq \widehat{\bigoplus_\kappa J_p}$ is pure. For any cardinal $\lambda = \lambda^{\aleph_0} > |R|$ there is a separable abelian $p$-group $G = \bigcup_{\alpha<\lambda} G_\alpha$ with a continuous chain of subgroups $\{G_\alpha\colon \alpha < \lambda\}$ with $G_0 = (0)$ such that the following hold:*

(i) $p^{\omega+1}(G/G_\alpha) = 0$ *for all* $\alpha < \lambda$.
(ii) *$G$ is not $p^\sigma$-projective for any ordinal $\sigma$. (Actually, $G$ is not fully starred.)*
(iii) $|G| = \lambda$.
(iv) $\operatorname{End} G = R + \operatorname{Small}^{[p]} G$.
(v) $R \cap \operatorname{Small}^{[p]} G = pR$.

Here $\operatorname{Small}^{[p]} G = \{\varphi \in \operatorname{End} G\colon \varphi(p^m G[p]) = 0 \text{ for some } m < \omega\}$ is the ideal of all endomorphisms $\varphi$ of $G$ that "act small" on the socle of $G$. As in the argument above, we can show that the groups $G$ in Theorem A are all efi: As the construction in the proof of Theorem A in [DG] shows, $G$ contains $\bigoplus_\lambda(\bigoplus_{n<\omega} R/p^n R)$ as a pure subgroup. Let $\pi$ be an idempotent in End $G$, $\pi = r + \sigma$, $r \in R$, $\sigma \in \operatorname{Small}^{[p]} G$. Then

$$\pi(p^m G[p]) = (p^m G[p])r \supseteq \bigoplus_\lambda (p^m R/p^{m+1} R)r.$$

Since $\pi(G)$ is countable and $\lambda$ uncountable, we conclude $r \in pR \subseteq \operatorname{Small}^{[p]} G$. Thus $\pi$ is an idempotent in $\operatorname{Small}^{[p]} G$ which implies that $\pi$ is bounded. Thus all countable summands of $G$ are bounded which implies that $G$ is efi.

In order to obtain non-thick groups $G$ in Theorem A we will prove

THEOREM B: *Same as Theorem A except that clause (ii) is replaced by the clause*

(ii)* *$G$ contains an unbounded $\sum$-cyclic subgroup $D$ and a subgroup $K$ such that $G = D + K$ and $D \cap K = D[p]$.*

It is easy to see that (ii)* implies that $G$ is not thick: Note that $G/K = (D+K)/K \cong D/D \cap K = D/D[p]$, the group $D^* = D/D[p]$ is unbounded and $\Sigma$-cyclic, and the natural map $\varphi\colon G \to D^*$ is not small since $(p^m D)[p^2]$ is not contained in $K = \ker\varphi$ for all $m < \omega$. The result in [BW] implies that $K[p]$ is not purifiable!

If $\pi = r + \sigma \in \operatorname{End} G = R + \operatorname{Small}^{[p]} G$ is an idempotent, then $r + pR$ is an idempotent in $R/pR$. If $R_n = \prod_{i=1}^n J_p$ is the (ring) cartesian product of $n$ copies of the $p$-adic integers, then idempotents lift modulo $pR$ and it is routine to verify that if $\operatorname{End} G_n = R_n + \operatorname{Small}^{[p]} G_n$ then $G_n$ has decomposition number $n$, i.e. $G_n$ decomposes into $n$ unbounded summands but every decomposition into $n+1$ summands involves a bounded one. On the other hand, if $R_\omega = \langle 1, \bigoplus_\omega J_p \rangle \leq \prod_\omega J_p$ and $G_\omega$ is as in Theorem B with $R = R_\omega$, then $G$ is efi but does not have finite decomposition number. This shows the existence of non-thick efi groups with and without finite decomposition number.

We will now recall the basic features of the construction in [DG] and indicate how to modify the proof to show Theorem B.

Let $R$ be a ring as in Theorem A and $T = {}^{\omega>}\lambda$ the tree of all functions $\tau: n \to \lambda$, $n < \omega$. Here we identify $n$ with $\{0, 1, \ldots, n-1\}$. $T$ carries a partial order $\leq$ defined by: $\sigma \leq \tau$ if $\operatorname{dom}\sigma \subseteq \operatorname{dom}\tau$ and $\sigma = \tau \restriction \operatorname{dom}\sigma$. Define $l(\tau) = n$ if $n = \operatorname{dom}\tau$. If $\tau \in T$, then we consider $\tau$ to be a generator of a cyclic $R$-module $\tau R \simeq R/p^{l(\tau)}R$. We set $B = \bigoplus_{\tau\in T} \tau R$. Then $B$ is a $\sum$-cyclic $p$-group and $\bar{B}$ denotes the torsion completion of $B$.

Each $g \in \bar{B}$ can be written as a convergent sum $g = \sum_{\tau\in T} \tau g_\tau$, with $g_\tau \in R$. The support of $g$ is defined to be

$$[g] = \{\tau \in T: g_\tau \notin p^{l(\tau)}R\}.$$

Note that for $g \in \bar{B}$, $[g]$ is at most countable and for each $m < \omega$, $\{\tau \in [g]: g_\tau \in p^m R\}$ is cofinite in $[g]$. Recall that a branch $b$ of $T$ is a maximal linearly ordered subset of $T$. Let

$$b(0) = \{\tau \in T \mid \tau(i) = 0 \text{ for all } i \in \operatorname{dom}\tau\}.$$

Then $b(0)$ is a branch and we call $b(0)$ the (constant) zero branch of $T$. Let

$$D = \bigoplus_{\tau\in b(0)} \tau R \cong \bigoplus_{n<\omega} R/p^n R.$$

Then $D$ is a natural summand of $B$. Finally we introduce a norm function $\| \ \|: (\operatorname{cf}(\lambda) + 1) \to (\lambda + 1)$, which is any fixed, continuous, strictly increasing function with $\|0\| = 0$ and $\|\mathrm{cf}(\lambda)\| = \lambda$. We extend this map to $T$ and subsets of $\bar{B}$: if $g \in \bar{B}$, then

$$\|g\| = \min\{\nu < \mathrm{cf}(\lambda): [g] \subseteq^{\omega>} \|\nu\|\} \quad \text{and} \quad \|x\| = \sup\{\|x\|: x \in X\}$$

for any $X \subseteq \bar{B}$. If $\|X\|$ is undefined, we say $\|X\| = \infty$; note that this can only happen if $|X| \geq \mathrm{cf}(\lambda)$. We are now able to describe how the groups $G$ in [DG] are defined: For some ordinal $\lambda^*$ of cardinality $\lambda$ the group $G$ is the union $G = \bigcup_{\alpha<\lambda^*} G_\alpha$ of a continuous chain of subgroups $G_\alpha$ where

$$G_0 = D, \quad G_1 = \langle D, b: b \in B, \ \|b\| < \|P_1\|\rangle.$$

(The $P_i$'s are the canonical submodules provided by the Black Box. All we need here and now is that $\|P_\alpha\| \leq \|P_\beta\|$ if $\alpha \leq \beta < \lambda^*$.) If $\mu$ is a limit ordinal, $\mu < \lambda^*$, we set $G_\mu = \bigcup_{\alpha<\mu} G_\alpha$.

Suppose $G_\alpha$, $\alpha < \lambda^*$ is already defined. Define

$$G_\alpha^* = \langle G_\alpha \cup \{b \in B: \|b\| < \|P_\alpha\|\}\rangle.$$

Note that $G_\alpha$ is a direct summand of $G_\alpha^*$ with $G_\alpha^*/G_\alpha$ $\Sigma$-cyclic. Next we pick a constant branch $k_\alpha = \{k_\alpha(i) | i < \omega\}$. Here we have $k_\alpha: \omega \to \{\nu_\alpha\}$ and $k_\alpha(i) = k_\alpha \restriction i$, $\|\nu_\alpha\| > \|P_\alpha\|$, $i = \mathrm{dom}\, k_\alpha(i)$. W.l.o.g. we may assume that $\nu_\alpha \notin \{\nu_\beta: \beta < \alpha\}$. The black box gives a branch $g_\alpha = \{g_\alpha(i): i < \omega\}$ and we set

$$a = \sum_{i<\omega} p^{l(g_\alpha(i))-1} g_\alpha(i).$$

Recall $\|g_\alpha\| = \|P_\alpha\|$. We introduce

$$y_n = \sum_{i \geq u} p^{l(k_\alpha(i))-n} k_\alpha(i) \qquad \text{for all } n < \omega.$$

Then $a \in \bar{B}[p]$ and $o(y) = p^n$. Moreover $\{y_n p^{n-1}: 0 < n < \omega\}$ is independent modulo $G_\alpha^*$ over $R$. Let $s \in \bar{B}[p]$ with $\|s\| < \|P_\alpha\|$ and $s_n \in \bar{P}_\alpha[p^{n+1}]$ such that

$$s - p^n s_n = b_n' \in G_\alpha[p] \cap B \quad \text{and} \quad \|s_n\| \leq \|s\| < \|\bar{P}_\alpha\|.$$

(Note that $s$, $s_n$ and $b_n'$ may all be 0.) Now define

$$g_{\alpha n} = a_n + y_n + s_n \quad \text{where} \quad a_n = \sum_{i=n+1}^{\infty} p^{l(g_\alpha(i))-n-1} g_\alpha(i).$$

Then

$$p^n g_{\alpha n} = (a - b_n) + 0 + (s - b_n') \quad \text{where} \quad b_n = \sum_{i=0}^{n} p^{l(g_\alpha(i))-n-1} g_\alpha(i).$$

We set $G_{\alpha+1} = \langle G^*_{\alpha'} \cup \{g_{\alpha n}: n < \omega\}\rangle$. As shown in [DG], the right choice of the elements $s$ will yield a group $G$ that satisfies all the conditions of Theorem A, except (ii). Note that even though $G_{\alpha+1}/(B \cap G_{\alpha+1})$ is *not* divisible, $G/B$ *is* divisible since $(G_{\alpha+1} + B)/B$ is divisible. Thus $G$ is pure in $\bar{B}$.

We will show that $G$ satisfies (ii)*. Define

$$K = D[p] + \bigoplus_{\tau \in T - b(0)} \tau R + \langle g_{\alpha n}: n < \omega, \alpha < \lambda^*\rangle_R$$

and

$$K_0 = D[p] + \bigoplus_{\tau \in T - b(0)} \tau R.$$

For $0 < \alpha < \lambda^*$ we define

$$K_\alpha = \langle D[p], \tau, g_{\beta n}: n < \omega,\ \beta < \alpha,\ \tau \in T, \|\tau\| < \|P_\alpha\|\rangle_R.$$

Obviously, $K = \bigcup_{\alpha < \lambda^*} K_\alpha$ and $G = D + K$ and $G_\alpha = D + K_\alpha$. By transfinite induction we will show $(I_\alpha)$: $D \cap K_\alpha = D[p]$ for $\alpha < \lambda^*$. Trivially, $(I_0)$ is true and we don't have to worry about limit ordinals $\alpha$. Now suppose $(I_\alpha)$ is true. Let $y_\alpha \in K_\alpha$ and suppose

$$x = y_\alpha + \sum_{n=1}^{N} g_{\alpha n} r \in K_{\alpha+1} \cap D.$$

Note that

$$x = y_\alpha + \sum_{n=1}^{N} (a_n + y_n + s_n) r_n$$

and that $\{y_n p^{n-1}: n \in \omega - \{0\}\}$ is independent modulo $\langle b \in \bar{B}: \|b\| \le \|P_\alpha\|\rangle_R$. This implies $p^n$ divides $r_n$ and thus $r_n = p^n t_n$ for some $t_n \in R$.

The relations between the elements involved now imply:

$$\begin{aligned}
x &= y_\alpha + \sum_{n=1}^{N} (a_n p^n t_n + y_n p^n t_n + s_n p^n t_n) \\
&= y_\alpha + \sum_{n=1}^{N} ((a - b_n) t_n + (s - b_n') t_n) \\
&= y_\alpha + a \sum_{n=1}^{N} t_n + s \sum_{n=1}^{N} t_n - \sum_{n=1}^{N} b_n t_n - \sum_{n=1}^{N} b_n' t_n \in D \cap K_{\alpha+1}.
\end{aligned}$$

Since the support of $a$, being the $\alpha^{\text{th}}$ branch provided by the black box, is almost disjoint to $[y_\alpha]$, and $[a] \cap [s]$ is finite as well since $\|s\| < \|a\| = \|P_\alpha\|$, we conclude $p \mid \sum_{n=1}^N t_n$, i.e. $a \sum_{n=1}^N t_n = 0$. Since $s \in \bar{B}[p]$, we obtain

$$x = y_\alpha - \sum_{n=1}^{N} b_n t_n - \sum_{n=1}^{N} b'_n t_n \in D \cap K_{\alpha+1}$$

and, in addition, $x \in K_\alpha$. Therefore $D \cap K_{\alpha+1} \subseteq K_\alpha \cap D \subseteq D \cap K_{\alpha+1}$ which shows $D[p] = K_\alpha \cap D = K_{\alpha+1} \cap D$. Moreover $D[p] = K \cap D$ since $K = \bigcup_{\alpha<\lambda^*} K_\alpha$.

This implies (ii)* in Theorem B. ∎

We would like to conclude by posing the following open

QUESTION: *Are there separable p-groups $A$ with a countable basic subgroup such that $A$ is* efi *but not thick?*

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